DIFFERENTIAL TRANSFORMATION METHOD FOR SEIR MEASLES MODEL

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ABSTRACT: *Present paper is based on the analysis of SEIR type models, which are used to study the transmission dynamics of infectious diseases in a population. Basic idea of disease transmission using compartmental modeling is discussed. A semi analytic technique Differential Transformation Method (DTM) is discussed in detail, which is used to compute an approximate solution of the non-linear system of differential equation governing the problem. A Measles SEIR model for Childhood diseases is solved and analyzed by Differential Transformation Method. DTM is proved to be very efficient and inexpensive technique for solving epidemic models as it gives better accuracy for small number of terms in a series solution. Moreover the method is easy to implement and involve less calculations. The Method has also been developed for SEIR model for measles dynamics.*

Key words: SEIR models, stability analysis, Differential Transformation Method (DTM*)*

INTRODUCTION

Epidemiologists use mathematical models to understand previous outbreaks of diseases and to better understand the dynamics of how infections spread through populations. Many existing models closely approximate historical disease patterns. Measles is a natural and fatal epidemic and it has become the calamity in our country. Measles is one of the best documented of human diseases, as far as population dynamics is concerned and is thus ideally suited to the testing of mathematical models and numerical methods developed for the solution of the model equations. An electronic literature search reveals numerous references to the relevant factors associated with measles models, including persistence of infection and mass vaccination, recurrence of epidemics in developed countries, seasonality and age structure [1, 2].

Bolker and Grenfell [2] examine both deterministic and stochastic models and discuss the effect of adding biological realism, in terms of age structure, on the tendency of a model to exhibit chaos. Epidemiology is one of the areas of the bio-medical sciences in which chaotic behavior is believed to be possible [3] it is one of the profusion of examples given in the popular book by Gleick [4]. In this paper SEIR model is discussed which is in the form of non-linear ODE's by using Differential Transformation method. The classical Taylor series method is one of the earliest analytic technique to many problems, specially ordinary differential equations. However, since it requires a lot of symbolic calculation for the derivatives of functions, it takes a lot of computational time for higher order derivatives. Here, we introduce the updated version of the Taylor series method which is called the differential transform method (DTM). The DTM is the method to determine the coefficients of the Taylor series of the function by solving the induced recursive equation from the given differential equation. The basic idea of the DTM was introduced by Zhou [5]

We used this method because it is efficient and gives us better accuracy rather than other numerical methods like RK-4 method. The numerical results give us in tabular form as well as in graphical form. The graphical results show that if we control the parameters of disease then we diseases would be decreases.

SEIR MEASLES MODEL

This model divides the total population into four compartments, one is susceptible, second is exposed, third is infective, and fourth one is recovered. Susceptible are those peoples who are not exposed to the disease, while the individuals who lead the exposed compartment are who catch the disease but are not able to spread this disease for a certain time of period (i.e incubation period) after moving on exposed compartment they move on infected compartment. Infective can pass the disease on, and last the recovered compartment contain those individuals who get recovery for a disease and get permanent immunity and those in the recovered group can no longer catch the infection

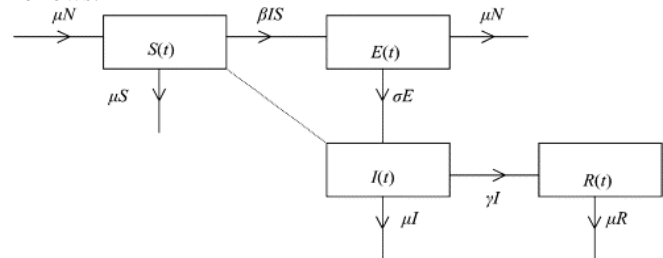The flow chart of SEIR measles model can be represented as follows:

Figure 1: Compartmental diagram for the SEIR model.Mathematical Form of the model

$$\frac{dS}{dt} = \mu - \beta SI - \mu S \quad t > 0$$

$$\frac{dE}{dt} = \beta SI - \mu e - \sigma e \quad t > 0$$

$$\frac{dI}{dt} = \sigma e - \mu i - \gamma i \quad t > 0$$

where

$$S(0) = S^0 \quad , \quad E(0) = E^0 \quad , \quad I(0) = I^0$$

We will assume the total population is constant for size N.
$i.e \ S + E + I + R = N$
We assume the death and birth rate is equal in this case
$i.e \quad \pi = \mu$
Where $S(t)$ is represent the susceptible peoples and are those individuals who catch the disease, $E(t)$ is represent the exposed individuals these are infected but not infectious, $R(t)$ is represent population who get permanent immunity or resistance.
$\mu, \sigma, \beta, \gamma$ are population reproduction and death rate, rate at which exposed individuals become infectious, infection rate and rate of recovery respectively.
The system is qualitatively analyzed by two ways
- Disease Free Equilibrium.
- Endemic Equilibrium.

DISEASE FREE EQUILIBRIUM

When naturally, the disease die out then the solution of the above system asymptotically approaches a disease free population or equilibrium is of the form

$$S = N , E = 0 , I = 0$$
$$i.e \ (S, E, I, R) = (N, 0, 0, 0)$$

The threshold result of this equilibrium is:

$$R_0 = \frac{\beta \sigma S}{(\mu + \sigma)(\mu + \gamma)}$$

The disease free equilibrium is locally stable if $R_0 < 1$ and unstable for $R_0 > 1$

Conversely , when $R_0 > 1$ then endemic equilibrium is stable.

When disease free equilibrium is unstable that is disease persist in the population then Endemic equilibrium takes the form.

$$(S^{\cdot}, E^{\cdot}, I^{\cdot}) = \left[\frac{(\mu+\sigma)(\mu+\gamma)}{\beta\sigma} , \frac{\mu N}{\mu+\sigma} - \frac{\mu(\mu+\gamma)}{\sigma\beta} , \frac{\mu\sigma N}{(\mu+\sigma)(\mu+\gamma)} - \frac{\mu}{\beta} \right]$$

DIFFERENTIAL TRANSFORMATION METHOD

The DT M, a transformation technique based on Taylor series expansion, offers a convenient means for obtaining analytic solutions of differential equations. Although this technique first applied in engineering domain by ZHOU in 1986 [6], and is commonly used for the solution of electric circuit problems, it seems to be largely unknown to the research community. After that Chen and HO [7] worked in this technique and further developed these techniques for partial differential equations.

This is a semi analytic technique and the solution of this technique is in the form of series also this method give closed form solution unlike other numerical methods. The proposed method is very cheap for calculation and easily applicable to ordinary differential equation such as linear and non-linear to find the exact and approximate solution.

Differential Transformation Method

A differential Transformation $U(k)$ of a function $u(x)$ is in the form.

$$U(k) = \frac{1}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{x=x_0}$$

Here, $u(x)$ = Original function
$U(k)$ = Transformed function
The Inverse of differential Transformation is defined as:

$$u(x) = \sum_{k=0}^{\infty} U(k)(x - x_0)^k$$

When x_0 is taken as zero , then the above original function $u(x)$ will be defined in the form of finite series and above function can be expressed in the form as:

$$u(x) = \sum_{k=0}^{\infty} U(k) x^k$$

$$u(x) = \sum_{k=0}^{\infty} U(k) \frac{x^k}{k!} \left[\frac{d^k u(x)}{dx^k} \right]_{x=x_0}$$

From the above equation we can see that the basic idea of DTM is based on Taylor series.

Some Fundamental Results of the one-dimensional transformation Method

Some fundamental results of the one-dimensional transformation Method are listed below [8-12]

Table 1: Basic Fundamental Results

Original function	Transformed function
$z(t) = a(t) \pm b(t)$	$Z(k) = A(k) \pm B(k)$
$z(t) = \beta a(t)$	$Z(k) = \beta A(k)$
$z(t) = \partial a(t)/\partial t$	$Z(k) = (k + 1)A(k + 1)$
$z(t) = \partial^s a(t)/\partial t^s$	$Z(k) = (k + 1)(k + 2) \dots$ $(m + s) \, A(k + s)$
$z(t) = u(t)v(t)$	$Z(k) = \sum_{p=0}^{k} U(k - l)V(l)$
$z(t) = \exp(\mu t)$	$Z(k) = \mu^k/k!$
$z(t) = t^m$	$Z(k) = \delta(k - m)$ $= \begin{cases} 1 \ , & k = m \\ 0 , & k \neq m \end{cases}$
$z(t) = \sin(\omega t + \alpha)$	$Z(k) = \frac{\omega^k}{k!} \sin\left(\frac{\pi}{2}k + \alpha\right)$
$z(t) = \cos(\omega t + \alpha)$	$Z(k) = \frac{\omega^k}{k!} \cos\left(\frac{\pi}{2}k + \alpha\right)$
$D[c]$	$\delta(k)$
$z(t) = \int_{u_0}^{u} a(t) dt$	$Z(m) = \frac{A(m-1)}{m} \ where$ $m \geq 1 , A(0) = 0$

APPLICATION

By using the fundamental result of DTM. We obtained the following recurrence relation to the system (1.1 - 1.3) with respect to time t we get

$$S(k + 1) = \frac{1}{(k+1)} \left[\mu\delta(k) - \beta \sum_{l=0}^{k} S(l)I(k - l) - \mu S(k) \right]$$

$$E(k + 1) = \frac{1}{(k+1)} \left[\beta \sum_{l=0}^{k} S(l)I(k - l) - \mu E(k) - \sigma E(k) \right]$$

$$I(k + 1) = \frac{1}{(k+1)} \left[\sigma E(k) - \mu I(k) - \gamma I(k) \right]$$

Now we start iterating the above system of equation for different values of parameters and with the help of these parameters, we will observe the effect of high or low vaccination on childhood disease, hence we discussed the stability.

Table 2: Simulation Results

Case	1	2	3	References
β	1×10^{-5}	0.5×10^{-5}	0.1×10^{-5}	[1]
μ	$0.02 per\ year$	$0.02 per\ year$	$0.02 per\ year$	[1]
σ	$45.6 per\ year$	$45.6 per\ year$	$45.6 per\ year$	[1]
γ	$73 per\ year$	$73 per\ year$	$73 per\ year$	[1]
N	50000000	50000000	50000000	[1]
$S(0)$	12500000	12500000	12500000	[1]
$E(0)$	50000	50000	50000	[1]
$I(0)$	30000	30000	30000	[1]

Case 1:

$R_0 = \frac{\beta\sigma s}{(\mu+\sigma)(\mu+\gamma)} = 1.7111 > 1$

Since $R_0 > 1\ i.e$

So the solution will converge to Endemic Equilibrium.

The tabular form and graphs are

s(t)= $12500000 - 3 \times 10^6 t - 51075 \times 10^2 t2 - 1258111450 \times t3 + 3.679753 \times 10^{10}t4 - 1.0229428 \times 10^{12}t5 + 2.3298723 \times 10^{13}t6 - 4.565351 \times 10^{14}t7 + 0.1463 \times 10^7 t7 - 0.6502 \times 10^9 t9 + \cdots$

$e(t) = 50000 + 1469 \times 10^3 t - 28370390t2 + 1689564564 \times t3 - 5.606072 \times 10^{10}t4 - 1.0229428 \times 10^{12}t5 + 2.3298723 \times 10^{13}t6 - 4.565351 \times 10^{14}t7 + 0.1463 \times 10 - 7t7 - 0.6502 \times 10 - 9t9 + \cdots$

$r(t) = 30000 + 894 \times 10^3 \times t + 30229206 \times t^2 - 1167008802 \times t^3 + 4.056478 \times 10^{10} \times t^4 - 1.10368191 \times 10^{12} \times t^5 + 2.50588823 \times 10^{13} \times t^6 - 1.8935676 \times 10^{14} \times t^7 + 3.39673378 \times 10^{15} \times t^8 - 5.9429815 \times 10^{16} \times t^9 + \cdots$
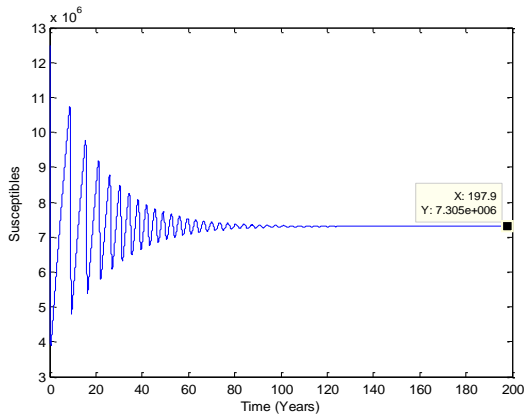
The graphs are:

Figure 1: This graph shows the relation between Susceptible and time in a years where
$\beta = 1 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year,\ \sigma = 45.6\ per\ year\ and\ \gamma = 73\ per\ year.$
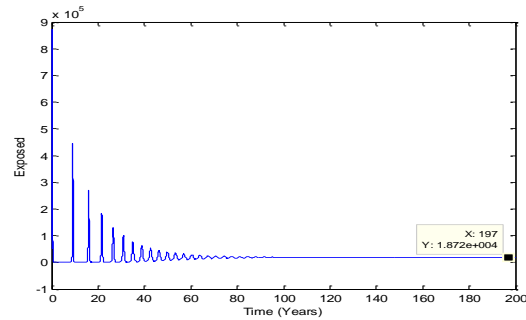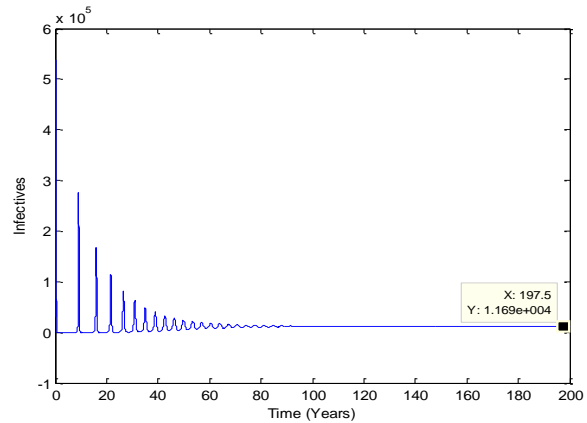

Figure 2: This graph shows the relation between exposed and time in a years where $\beta = 1 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year\ and\ \gamma = 73\ per\ year$.

Figure 3: This graph shows the relation between infected and time in a years where $\beta = 1 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year\ and\ \gamma = 73\ per\ year$.

Case 2: $\beta = 0.5 \times 10^{-5}$

$R_0 = \frac{\beta\sigma s}{(\mu+\sigma)(\mu+\gamma)} = 3.4222 > 1$

So the solution will converge to Endemic Equilibrium

s(t)= $12500000 - +375 \times 10^3 t - 60187.5 \times t^2 + 194909228.1 \times t^3 - 5619539471 \times t^4 + 1.0643475 \times 10^{11}t5 - 1.58535931 \times 10^{12}t6 + \cdots$

$e(t)$=$50000 - 1906 \times 10^3 t + 43532297.5t^2 -$
$856889964.2 \times t^3 + 1.5391395 \times 10^{10}t^4 +$
$2.4684337 \times 10^{11}t^5 - 2.9182787 \times 10^{11}t^6 + \cdots$
$i(t)$= $30000 + 894 \times 10^3 \times t - 46720794 \times t^2 +$
$1798875048 \times t^3 - 4.260701 \times 10^{10} \times t^4 +$
$7.62602290 \times 10^{11} \times t^5 - 7.40486029 \times 10^{12} \times t^6 + \cdots$
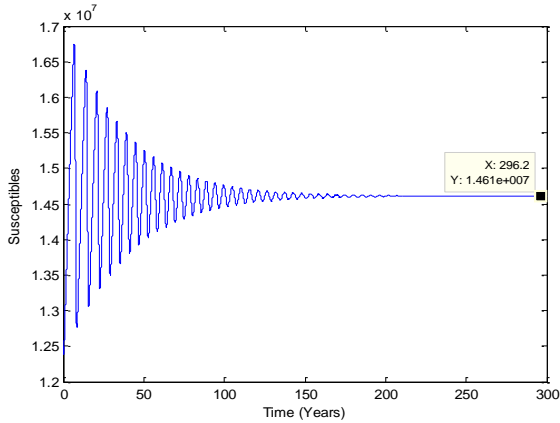The graphs are:

Figure 4: This graph shows the relation between susceptible and time in a years where $\beta = 0.5 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year$ and $\gamma = 73\ per\ year$.
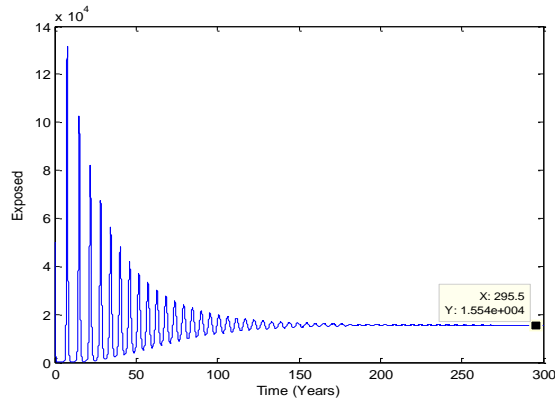

Figure 5: This graph shows the relation between exposed and time in a years where $\beta = 0.5 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year$ and $\gamma = 73\ per\ year$.
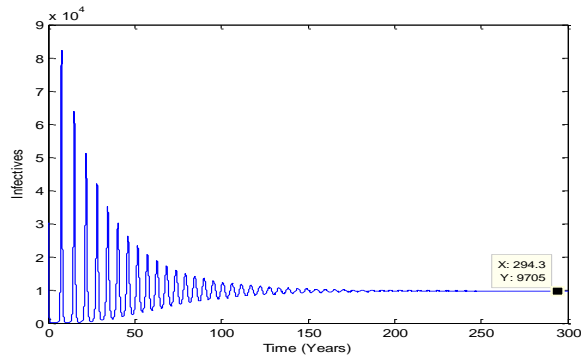

Figure6: This graph shows the relation between infected and time in a years where $\beta = 0.5 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year$ and $\gamma = 73\ per\ year$.

Disease Free Equilibrium of SEIR Model
Case 3: $\beta = 0.1 \times 10^{-5}$

$$R_0 = \frac{\beta \sigma s}{(\mu + \sigma)(\mu + \gamma)} = 0.6844 < 1$$

So the solution will converge to Disease Free equilibrium.

$s(t)$=$12500000 - 1125 \times 10^3 t - 2698125 \times t^2 +$
$261170393.8 \times t^3 - 7633150874 \times t^4 + 1.74962719 \times$
$10^{11}t^5 - 3.33900469 \times 10^{12}t^6 + 5.455373482 \times 10^{13}t^7 + \cdots$
$e(t)$=$50000 - 406 \times 10^3 t + 11970235t^2 - 443179779.5 \times t^3 +$
$1.26863104 \times 10^{10}t^4 - 2.906820823 \times 10^{11}t^5 +$
$5.548574244 \times 10^{12}t^6 - 9.07050458 \times 10^{13}t^7 + \cdots$
$i(t)$= $30000 + 894 \times 10^2 \times t - 46720794 \times t^2 + 1798875048 \times$
$t^3 - 4.260701 \times 10^{10} \times t^4 + 7.62602290 \times 10^{11} \times t^5 -$
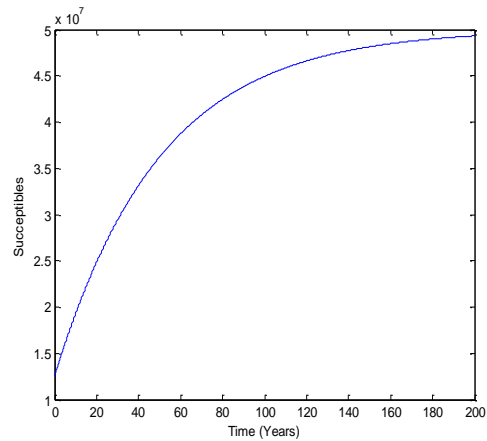$7.40486029 \times 10^{12} \times t^6 + \cdots$
The graphs are:

Figure 7:This graph shows the relation between susceptible and time in a years where $\beta = 0.1 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year$ and $\gamma = 73\ per\ year$.
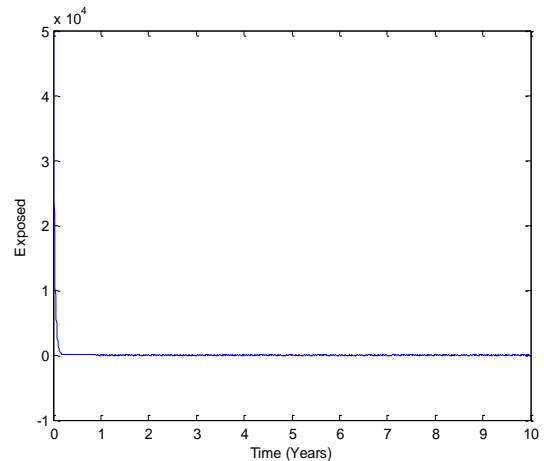

Figure 8: This graph shows the relation between exposed and time in a years where $\beta = 0.1 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year$ and $\gamma = 73\ per\ year$.
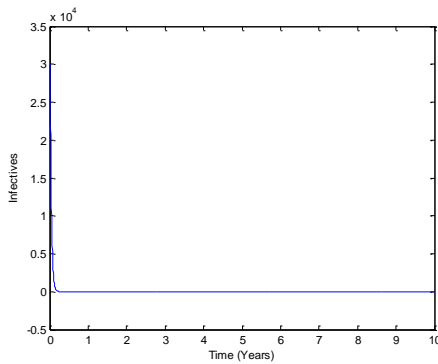
Figure 9: This graph shows the relation between infected and time in a years where $\beta = 0.1 \times 10^{-5}, N = 50000000, \mu = 0.02\ per\ year, \sigma = 45.6\ per\ year\ and\ \gamma = 73\ per\ year$.

CONCLUSION

A semi analytic technique, Differential Transformation Method (DTM) is applied in SEIR Measles Model to solve and analyze a vaccination. Many researchers apply many mathematical numerical techniques to solve the measles models. To analyze SEIR model, three different cases for different values of contact rate β are discussed and plot the number of Susceptible `S', Exposed `E' Infectious individuals `I' against time in years, in each case. For this purpose we are trying to control the parameters by using mathematical model with very efficient technique DTM. The reason to use Differential Transformation Method it gives better accuracy for small number of terms it has really proved to be a method in which there is a less probability of occurrence of error.

We developed DTM for SEIR model measles dynamics. We discussed three different cases for the different values of contact rate β .It is clear from graphs drawn for the three cases that in 2nd case the solution will approach to disease free equilibrium and in case 1st and 3rd the solution will approach to endemic equilibrium. It is concluded that by decreasing the contact rate β the value of R_0 will be decrease and hence we can achieve the disease free stage by decreasing the contact rate β. By the isolation of the infections individuals from the other ones, the spread of disease into a population can be controlled as the contact rate β will be decreased. For this purpose, appropriate measures should be taken so that the interaction between infectious and susceptible children should be minimized. Moreover, the analysis of epidemic models using Differential Transformation Method reveals that, the method provides rapidly convergent series solution by little iteration and avoids the massive computational work. The method is very easy to implement and gives a better accuracy for a small number of terms.

By the graphical representation of these parameters we have shown that the goal more we create the awareness to the causes of the problem and opportunity prevention through vaccination. The number of death rates can be controlled as the awareness and if the campaign of vaccination rises the disease and its destruction falls down. Also we see that the infected people when vaccinated through gone under treatment will develop at immunity and will become healthy enough to fight against disease. On the other hand those who don't develop immunity will expire.

Lastly we conclude that we have to emphasis the fact of vaccination and its awareness among the masses is an important factor which can play a vital role in the development of the healthy society. We also concluded that by the isolation of the infected individuals from the other one we control the spread of disease into a population as the contact rate β will be decrease.

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